Comment on energy level statistics in the mixed regime

1997 J. Phys. A: Math. Gen. 30 8787

(http://iopscience.iop.org/0305-4470/30/24/037)

J. Phys. A: Math. Gen. **30** (1997) 8787–8793. Printed in the UK PII: S0305-4470(97)85316-4

COMMENT

Comment on energy level statistics in the mixed regime

Marko Robnik†§ and Tomaž Prosen‡‖

† Center for Applied Mathematics and Theoretical Physics, University of Maribor, Krekova 2, SLO-2000 Maribor, Slovenia

‡ Department of Physics, Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SLO-1111 Ljubljana, Slovenia

Received 24 June 1997, in final form 23 September 1997

Abstract. We comment on the recent paper by Abul-Magd (1996 *J. Phys. A: Math. Gen.* **29** 1) concerning the energy level statistics in the mixed regime, i.e. such having the mixed classical dynamics where regular and chaotic regions coexist in the phase space. We point out that his basic assumption on the additive property of the level-repulsion function $r(S)$ (conditional probability density) in the sense of dividing it linearly into the regular and chaotic part in proportion to the classical fractional phase space volumes ρ_1 and $\rho_2 = q$ is *not justified*, since among other things, it relies on the type of Berry's ergodic assumption, which, however, is only correct in a homogeneous ensemble of ergodic systems, not in the neighbourhood of an integrable system. Thus, his resulting distribution cannot be regarded as a theoretically well-founded object. We point out that the semiclassical limiting energy level spacing distribution must be of Berry–Robnik (1984) type, and explain what transitional behaviour of the Brody-type (with fractional power-law energy level repulsion) we observe in the near semiclassical regime where effective $\hbar$ is not yet small enough. Thus we refer to the derivation, arguments and conclusions in our previous paper (Prosen and Robnik 1994 *J. Phys. A: Math. Gen.* **26** 8059), and explain again the behaviour in this double transition region.

Abul-Magd (1996) recently offered a new theoretical energy level spacing distribution for quantal Hamiltonian systems whose classical dynamics is of the mixed type, i.e. such having regular regions of invariant tori coexisting in the phase space (and on the energy surface) with chaotic regions, a typical KAM scenario. In this comment we wish to point out that his result is not theoretically well founded and is in fact erroneous, and has no other merit than mathematical simplicity, which, however, is of course not a sufficient condition for the scientific merit. Abul-Magd used the famous Wigner surmise (Wigner 1956, Brody 1973, Brody *et al* 1981, Robnik 1984, Bohigas and Giannoni 1984), which by itself is a sound argument, but Abul-Magd made an assumption about the conditional probability density $r(S)$ (the so-called level repulsion function), in conjunction with the Berry-type argument on the ergodicity of quantal energy spectra in an ensemble of classically ergodic systems (Berry 1981, 1983, 1985), which is wrong in his context, because it is applied to the systems that are *not* ergodic but close to an integrable system (KAM-type systems).

Quite generally, by knowing $r(S)$ one gets the level spacing distribution $P(S)$ at once as $P(S) = r(S)\exp(-\int_0^S r(x)\,\mathrm{d}x)$. For example, $r(S) = 1$ implies Poisson distribution,

§ E-mail address: robnik@uni-mb.si
‖ E-mail address: prosen@fiz.uni-lj.si

$r(S) = \pi S/2$ implies Wigner (two-dimensional GOE), $r(S) \propto S^\beta$ implies Brody distribution etc.

In order to derive $r(S)$ Abul-Magd referred to the ergodicity argument by Berry (1981, 1983, 1985) where in calculating the $P(S)$ at small S he (Berry) replaced the average over the energy spectrum by the average over an ensemble of *classically ergodic* systems parametrized by at least two parameters (because the degeneracies, the diabolical points, have codimension 2), which is a very reasonable assumption indeed, and it immediately yields the linear level repulsion. However, as it has been pointed out by one of us (Robnik 1984) this ergodicity assumption cannot be applied in the neighbourhood of an integrable system, simply because there is no local uniformity in the parameter space, and as we approach (the coordinates/parameters of) the integrable system we see greater and greater density of degeneracy points: they are not uniformly distributed in the space {parameter Ax parameter Bx energy}.

Therefore, the ansatz ('the basic assumption') of Abul-Magd for $r(S) = \rho_1 + \rho_2 S$, where $\rho_1 + \rho_2 = 1$ (in his notation $q \equiv \rho_1$), is not justified theoretically, but is just a guess. In fact it leads to a distribution function which is mathematically simple, normalized, but not its first moment, which is another deficiency of the model.

Further, we claim with full theoretical justification that the correct ultimate semiclassical energy level spacing distribution is in fact of Berry and Robnik (1984). (Similar thinking can be of course applied to other statistical measures, such as number variance and delta statistics, etc, see e.g. Seligman and Verbaarschot (1985).) This assertion has a sound theoretical foundation. It is based on the picture in the quantum phase space (the Wigner functions of stationary eigenstates) in the strict semiclassical limit, $\hbar \to 0$, where we observe the condensation of states in volume elements of order $(2\pi\hbar)^f$, where f is the number of degrees of freedom (see e.g. Robnik 1988, 1997), on classical invariant objects, which is the contents of the so-called principle of uniform semiclassical condensation. The prediction agrees with the rigorous results by Lazutkin (1981, 1991) on splitting the energy spectra and the eigenstates in regular and irregular levels/states (qualitatively predicted by Percival (1973)), in the special case of convex billiards with smooth boundaries. This has also been analysed in Li and Robnik (1995). We have at least two special but typical mixed dynamical systems for which we have demonstrated with a very great accuracy that the semiclassically limiting statistics is Berry–Robnik.

We show the results in figures 1(a)–(d), for two different representations of the level spacing distribution $P(S)$. Namely, we show the diagrams of cumulative level spacing distribution $W(S) = \int_0^S P(x)\,dx$, and the so-called U-function, defined by $U(W) = \frac{2}{\pi}\arccos\sqrt{1-W}$ (Prosen and Robnik 1993). In figures 1(a) and (b) we show the results for the quantized compactified standard map exactly as published in Prosen and Robnik (1994a, b), but now we also include the best-fitting Abul-Magd curves. The system is a little bit abstract but it allows us to achieve the deepest possible semiclassical regime with strongly significant statistics. In figures 1(c) and (d) we show the results for a generic physical autonomous two-dimensional Hamiltonian system, namely the semiseparable harmonic oscillator exactly as in Prosen (1995, 1996), but now we also include the best fitting Abul-Magd curves. In both cases the Berry–Robnik best-fitting curve is statistically significant, in contradistinction to the Abul-Magd best-fitting curves. The quantitatively derived Berry–Robnik ρ_1 agrees with the classical one within 1–3%, whilst the Abul-Magd parameter q is in fact almost twice as large as the classical ρ_1, which is another *a posteriori* reason to reject his derivation and interpretation. It is interesting to note that in analysing the numerical spectra we had to use the infinitely dimensional GOE statistics on chaotic component (Wigner

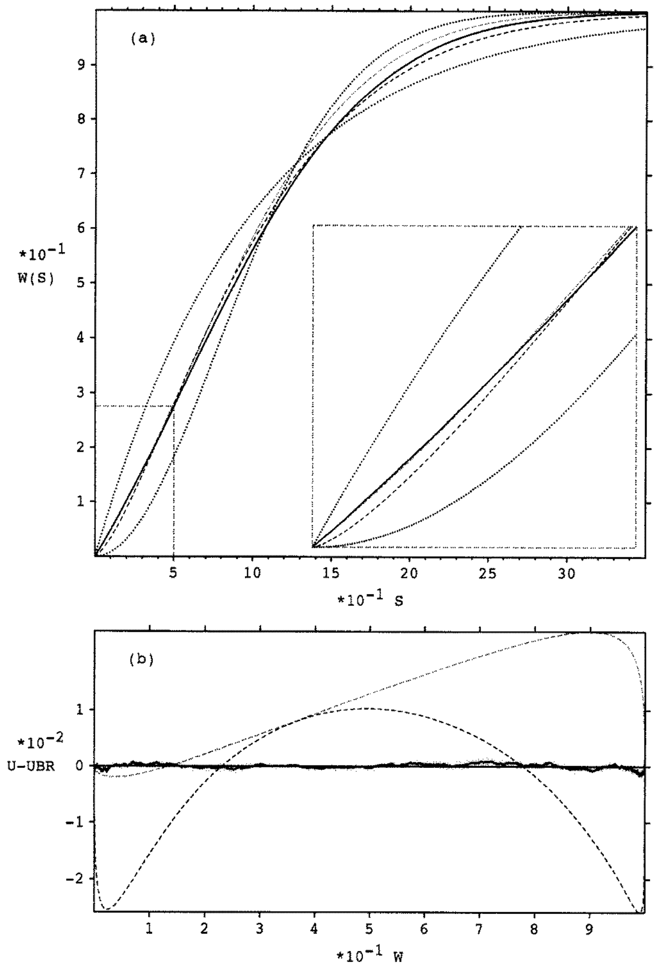

Figure 1. (a), (c) show the results for $W(S)$ and (b)–(d) show the so-called U-function $U(W) - U(W_{\rm BR})$. Here $W_{\rm BR}$ refers to the best fitting Berry–Robnik level spacing distribution, so that abscissa in the diagrams (b), (d) is the ideal agreement with Berry–Robnik statistics. The results for the quantized compactified standard map are in (a), (b) and for the two-dimensional semiseparable autonomous Hamiltonian harmonic oscillator in (c), (d). The full heavy curve is data, the full light curve is the best-fitting Berry–Robnik, the broken curve is best-fitting Brody and the chain curve is the best-fitting Abul-Magd. Abul-Magd is the upper curve and Brody is the lower one. It is clearly seen for big S in the W-plots that the disagreement with Abul-Magd's prediction is very bad on this global scale, and this disagreement turns out to be indeed very big in the U-function plots, except perhaps at small S. For the reference we plot here also the $\pm\sigma$ bands (grey) of expected statistical standard deviation. For the sake of completeness we quote the best fitting parameter values: In (a), (b) we have the classical $\rho_1 = 0.265$, the quantal Berry–Robnik $\rho_1 = 0.273$ and the quantal Abul-Magd $q = 0.448$. In (c), (d) we have the classical $\rho_1 = 0.291$, the quantal Berry–Robnik $\rho_1 = 0.286$ and the quantal Abul-Magd $q = 0.466$. In (a), (b) we have 160 000 numerical quasi-energy levels for quantum maps with dimensions 15 982–16 000, with the same kick parameter $a = 1.8$ and the same classical limit. In (c), (d) we have a stretch of 13 445 energy levels starting from around 17 684 000th level. In plots (a) and (c) we show for comparison also the GOE and Poissonian curves (dotted), and in the inset the magnification of the situation at small spacings S. In (a) the differences between the data and theory (Berry–Robnik) are not visible, whilst in (c) they can be seen, especially in the inset, whilst in both (b) and (d) the (quite small) differences between the data and theory (Berry–Robnik) are made visible.

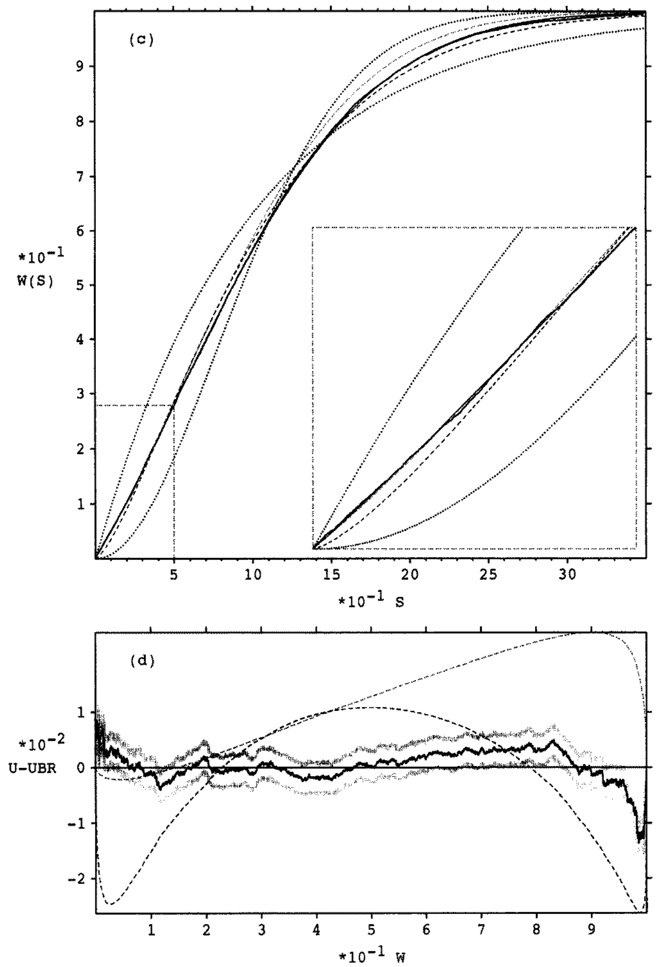

Figure 1. (Continued)

distribution = two-dimensional GOE was not good enough) in order to achieve perfect agreement between the numerical results and the best fitting Berry–Robnik distribution.

Therefore we have full confidence in the correctness of the asymptotic far semiclassical limit ($\hbar \to 0$) of spectral statistics.

However, before reaching the ultimate semiclassical limit, in a regime which we call the near semiclassical limit, we find phenomenologically significant and to some extent universal statistical behaviour of energy spectra, especially in two-dimensional billiards and elsewhere (Prosen and Robnik 1993, 1994a, b). Namely, we typically observe the fractional power law level repulsion, $P(S) \propto S^{\beta}$, where the exponent β can be anything between 0 and 1 for OE statistics, or $\beta \in [0, 2]$ in case of UE (broken antiunitary symmetries, or more generally, complex representations (see Robnik 1986, Leyvraz *et al* 1996, Keating and Robbins 1997, Dobnikar 1996, Robnik and Dobnikar 1997)). It is now qualitatively understood that this statistics is another manifestation of quantum (dynamical) localization, i.e. the localization of quantum eigenstates related to the classical dynamics. In KAM systems we have a theory on β, where we derive (Prosen and Robnik 1994b) the scaling law $\beta = \text{constant}\,\overline{h}$, for sufficiently small $\overline{h}$. Furthermore, we have shown that the fractional power-law regime with the given β should be observed for spacings S within the interval $[\exp(-1/\beta), 1]$. Using the above scaling estimates, we see that the fractional power-law level repulsion is *not* observed in the exponentially small interval $[0, \exp(-\text{constant}/\overline{h})]$. So, when the *effective* Planck constant goes to zero, $\overline{h} \to 0$, this interval becomes exponentially small and practically invisible, because there are usually not enough objects there. The estimate agrees perfectly well with the prediction by Berry and Robnik (1984) that there is such an exponentially small region at small S due to the tunnelling phenomena. Therefore, the picture is now fully consistent, and it remains to explain what behaviour we predict theoretically in this exponentially small region.

We know from elementary thinking that in this region the $P(S)$ must behave linearly $P(S) \propto S$, which cannot be predicted semiclassically (Robnik 1986, Berry 1991, Robnik and Salasnich 1997) but only quantally (Robnik 1987). The reason is that for very small spacings the quantum degenerate two-dimensional perturbation theory must be ultimately sufficient, which was demonstrated and argued in Robnik (1987). Indeed, if one increases the dimensionality of such a model ('Poissson + GOE'), one finds the same linear level repulsion law for three and four dimensions (Izrailev 1993) and for higher dimensions (Prosen 1993). This quantum-mechanical picture explains the linear level repulsion region, which is exponentially small.

The question then is to explain how—in this doubly transitional regime: mixed dynamics, and transition from near to far semiclassics—the Brody-like behaviour goes over into Berry–Robnik behaviour, as the $\overline{h}$ tends to 0. For this we have no global quantitative theory, except for the more or less local features described above. Schematically we show this in figure 2. We also show in figure 3, schematically, the Brody-like distribution and the Berry–Robnik distribution, with the indicated (and schematically exaggerated) exponentially small region of linear level repulsion (the purely quantum regime). In practice, with actual spectra, it is almost impossible to detect the exponentially small region, and indeed this has not been observed until now in any specific system.

The existence of the fractional power-law level repulsion and Brody-like behaviour is definitely connected with the existence of (dynamical) localization, which is a topic of current research. Moreover, in ergodic systems, but with very slow diffusion, we also observe dynamical localization (Prosen and Robnik 1994b, Borgonovi *et al* 1996, Frahm and Shepelyanski 1997, Casati and Prosen 1997, Robnik *et al* 1997), which gives rise to

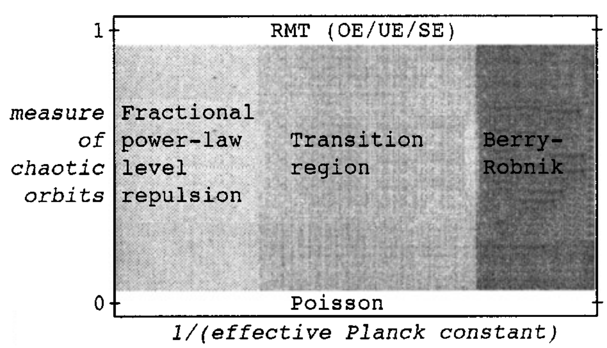

Figure 2. We show the schematic diagram of the doubly transition region: from integrable to ergodic classical dynamics and from near semiclassics (not very small $\overline{h}$) to far semiclassics (sufficiently small $\overline{h}$).

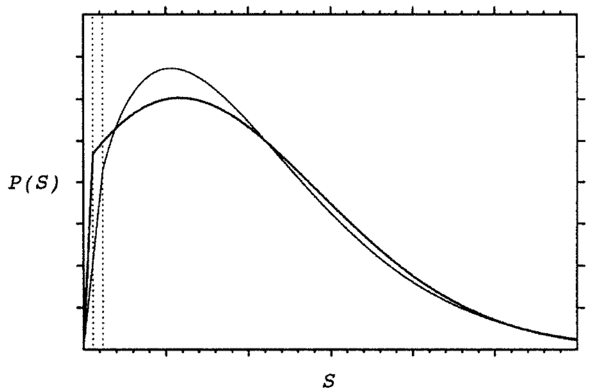

Figure 3. We show schematically two examples of the Brody-like level spacing distribution (with higher maximum) and Berry–Robnik type, but in both cases indicated the exponentially small (but here exaggerated) regime of linear level repulsion (see text).

the Brody-like behaviour, with fractional power-law level repulsion, but here the picture is much more complicated, and β must tend to 1, rather than 0, as $\hbar \to 0$.

Finally, in regard of phenomenological formulae, we suggest that the so-called Berry–Robnik–Brody (BRB) distribution (see Prosen and Robnik 1994b), which is a two-parameter distribution function, is the best, because it has the theoretical foundation in the sense that it takes into account the division of the classical phase space (parameter ρ_1), and the localization of the chaotic states on the (subset of the) chaotic regions (whose measure is $\rho_2 = 1 - \rho_1$), captured by the level repulsion parameter β. Indeed, in our own work (Prosen and Robnik 1994b) we have confirmed the agreement in billiard systems and in mappings, and a similar success is reported in the context of theoretical nuclear spectra by Lopac *et al* (1996).

In conclusion, we propose that there is no place for other limiting semiclassical energy level statistics than Berry–Robnik (1984), in systems with mixed classical dynamics (KAM-type systems), while in the transition regime there is evidence and substantial understanding that outside the exponentially small region of linear level repulsion due to tunnelling, there is the fractional power-law level repulsion and Brody-like behaviour with exponent $\beta = \text{constant}\,\hbar$, which goes to zero when $\hbar$ goes to zero, thereby going over to the Berry–Robnik distribution. We have explained why the basic assumption of Abul-Magd (1996) is not justified and therefore any significant agreement of his results with high-quality spectral data cannot be expected. Indeed, this has been clearly demonstrated in figures 1(*a*)–(*d*).

Acknowledgments

We would like to thank the Ministry of Science and Technology of the Republic of Slovenia for financial support. This work was also supported by the Rector's Fund of the University of Maribor.

References

Abul-Magd A Y 1996 *J. Phys. A: Math. Gen.* **29** 1
Berry M V 1981 *Ann. Phys., NY* **131** 136

——1983 *Chaotic Behaviour of Deterministic Systems* ed G Iooss, R H G Helleman and R Stora (Amsterdam: North-Holland) p 171
——1985 *Chaotic Behaviour in Dynamical Systems* ed G Casati (New York: Plenum)
——1991 *Chaos in Quantum Physics* ed M-J Giannoni, A Voros and J Zinn-Justin (Amsterdam: North-Holland)
Berry M V and Robnik M 1984 *J. Phys. A: Math. Gen.* **17** 2413
Bohigas O and Giannoni M-J 1984 *Lecture Notes in Physics* vol 209 (Berlin: Springer) p 1
Borgonovi F, Casati G and Li Baowen 1996 *Phys. Rev. Lett.* **77** 4744
Brody T A 1973 *Lett. Nuovo Cimento* **7** 482
Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M 1981 *Rev. Mod. Phys.* **53** 385
Casati G and Prosen T 1997 *Phys. Rev. Lett.* submitted
Dobnikar J 1996 *Diploma Thesis* CAMTP, unpublished (in Slovenian)
Frahm K and Shepelyanski D 1997 *Phys. Rev. Lett.* **78** 1440
Izrailev F M 1993 Private communication
Keating J P and Robbins J M 1997 *J. Phys. A: Math. Gen.* **30** L177
Lazutkin V F 1981 *The Convex Billiard and the Eigenfunctions of the Laplace Operator* (Leningrad: University Press) (in Russian)
——1991 *KAM Theory and Semiclassical Approximations to Eigenfunctions* (Heidelberg: Springer)
Leyvraz F, Schmit C and Seligman T H 1996 *J. Phys. A: Math. Gen.* **29** L575
Li Baowen and Robnik M 1995 *J. Phys. A: Math. Gen.* **28** 4483
Lopac V, Paar H and Brant S 1996 *Z. Phys.* A **356** 113
Percival I C 1973 *J. Phys. B: At. Mol. Phys.* **6** L229
Prosen T 1993 unpublished
——1995 *J. Phys. A: Math. Gen.* **28** L349
——1996 *Physica* **91D** 244
——1997a *Phys. Lett.* A **233** 323
——1997b *Phys. Lett.* A **233** 332
Prosen T and Robnik M 1993 *J. Phys. A: Math. Gen.* **26** 2371
——1994a *J. Phys. A: Math. Gen.* **27** L459
——1994b *J. Phys. A: Math. Gen.* **27** 8059
Robnik M 1984 *J. Phys. A: Math. Gen.* **17** 1049
——1986 *Lecture Notes in Physics* vol 263 (Berlin: Springer) p 120
——1987 *J. Phys. A: Math. Gen.* **20** L495
——1988 *Atomic Spectra and Collisions in External Fields* ed K T Taylor, M H Nayfeh and C W Clark (New York: Plenum) pp 265–74
——1997 *Open Syst. Inf. Dynam.* **4** 211–40
Robnik M and Dobnikar J 1997 to be published
Robnik M, Dobnikar J, Liu Junxian and Veble G 1997 work in progress
Robnik M and Salasnich 1997 *J. Phys. A: Math. Gen.* **30** 1711
Seligman T H and Verbaarschot J J M 1985 *J. Phys. A: Math. Gen.* **18** 2227
Wigner E P 1956 *Conference on Neutron Physics by Time-of-Flight (Galinburg, TN)* reprinted in: Porter C E (ed) 1965 *Statistical Theories of Spectra: Fluctuations* (New York: Academic)